The Principle of Differential Subordination and Its Application to Integral Operators

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Introduction

- Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

- For $a \in \mathbb{C}$ and a nonnegative integer n, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

- Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [12])

$$f \prec F \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

[12] S. S. Miller and P. T. Mocanu, Classes of univalent integral operators, *J. Math. Anal. Appl.* **157**(1991), 147-165.

Definition 1.1 (Miller and Mocanu [13])

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

[13] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

Definition 1.2 (Miller and Mocanu[14])

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

[14] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* **48**(2003), 815-826.

Definition 1.3 (Miller and Mocanu [14])

We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that

$$f'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \backslash E(f)).$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

which are analytic in $\mathbb{U}$ and let $\mathcal{A}_1 = \mathcal{A}$. We also denote the class $\mathcal{D}$ by

$$\mathcal{D} := \{\varphi \in \mathcal{H} : \varphi(0) = 1 \quad \text{and} \quad \varphi(z) \neq 0 \quad (z \in \mathbb{U})\}.$$

Let $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ consisting of all functions which are, respectively, starlike in $\mathbb{U}$ and convex in $\mathbb{U}$(see, for details, [13]).

Now we introduce the following integral operator $I^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ defined by

$$I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z) := \left(\frac{p\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z t^{\delta-1}f^{\alpha}(t)\varphi(t)dt\right)^{1/\beta} \tag{1.3}$$

$(f \in \mathcal{A}_p;\ \alpha,\gamma,\delta \in \mathbb{C};\ \beta \in \mathbb{C}\backslash\{0\};\ p\alpha+\delta = p\beta+\gamma;\ \Re\{p\alpha+\delta\} > 0;\ \phi,\varphi \in \mathcal{D})$.

[13] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[14] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* **48**(2003), 815-826.

Lemma 1.1 (Miller and Mocanu [10])

Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the following condition:*

$$\Re\{H(is, t)\} \leq 0,$$

for all real s *and* $t \leq -n(1 + s^2)/2$*, where* n *is a positive integer. If the function*

$$p(z) = 1 + p_n z^n + \cdots$$

is analytic in $\mathbb{U}$ *and*

$$\Re\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

[10] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* **28**(1981), 157-171.

Lemma 1.2 (Miller and Mocanu [11])

Let $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If*

$$\Re\{\beta h(z) + \gamma\} > 0 \quad (z \in \mathbb{U}),$$

then the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ *and satisfies the following inequality*

$$\Re\{\beta q(z) + \gamma\} > 0 \quad (z \in \mathbb{U}).$$

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[11] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, *J. Different. Equations* **567**(1985), 297-309.

Lemma 1.3 ((Miller and Mocanu [13])

Let $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{U}$ *with*

$$q(z) \not\equiv a \quad and \quad n \geq 1.$$

If q *is not subordinate to* p*, then there exist points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad and \quad \zeta_0 \in \partial\mathbb{U} \backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}),\ q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

[13] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

Let $c \in \mathbb{C}$ with $\Re\{c\} > 0$ and let

$$N := N(c) = \frac{|c|\sqrt{1 + 2\Re\{c\}} + \Im\{c\}}{\Re\{c\}}.$$

If $R(z)$ is the univalent function defined in $\mathbb{U}$ by $R(z) = 2Nz/(1 - z^2)$, then the open door function defined by

$$R_c(z) := R\left(\frac{z + b}{1 + \bar{b}z}\right) \quad (z \in \mathbb{U}), \tag{1.4}$$

where $b = R^{-1}(c)$[13].

[13] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

Remark 1.1

The function R_c defined by (1.4) is univalent in $\mathbb{U}$, $R_c(0) = c$ and $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane with slits along the half-lines $\Re\{w\} = 0$ and $|\Im\{w\}| \geq N$.

Lemma 1.4 (Miller and Mocanu [13])

Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $p\alpha + \delta = p\beta + \gamma$, $\Re\{p\alpha + \delta\} > 0$ *and* $\phi, \varphi \in \mathcal{D}$. *If* $f \in \mathcal{A}_{\varphi;p\alpha,\delta}$, *where*

$$\mathcal{A}_{\varphi;p,\alpha,\delta} := \left\{ f \in \mathcal{A}_p : \ \alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{p\alpha+\delta}(z) \right\} \qquad (1.5)$$

and $R_{p\alpha+\delta}(z)$ *is defined by* (1.4) *with* $c = p\alpha + \delta$*, then* $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta} \in \mathcal{A}_p$*,* $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)/z^p \neq 0$ *and*

$$\Re\left\{\beta\frac{z(I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z))'}{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma\right\} > 0 \quad (z \in \mathbb{U}),$$

where $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}$ is the integral operator defined by (1.3).

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ when $0 \leq s < t$.

Lemma 1.5 (Miller and Mocanu [14])

Let $q \in \mathcal{H}[a,1]$ *and let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$*. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \quad (z \in \mathbb{U}).$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$*, then*

$$h(z) \prec \varphi(p(z), zp'(z)).$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$*, then* q *is the best subordinant.*

Lemma 1.6 (Pommerenke [19])

The function

$$L(z,t) = a_1(t)z + \cdots$$

with

$$a_1(t) \neq 0 \quad and \quad \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot;t)$ *ia analytic in* $\mathbb{U}$ *for all* $t \geq 0$, $L(z;\cdot)$ *is continuously differentiable on* $[0,\infty)$ *for all* $z \in \mathbb{U}$. *If* $L(z;t)$ *satisfies*

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty)$$

and

$$|L(z;t)| \leq K_0|a_1(t)| \quad (|z| < r_0 < 1;\ 0 \leq t < \infty))$$

for some positive constants K_0 *and* r_0, *then* $L(z;t)$ *is a subordination chain.*

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[19] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck and Ruprecht, Göttingen, 1975.

Subordination theorem involving the integral operator $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}$ defined by (1.3) is contained in Theorem 2.1 below.

Theorem 2.1

Let $f, g \in \mathcal{A}_{\varphi;p,\alpha,\delta}$, where $\mathcal{A}_{\varphi;p,\alpha,\delta}$ is defined by (1.5). Suppose also that

$$\Re\left\{1+\frac{z\nu''(z)}{\nu'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U};\ \nu(z) := z\left(\frac{g(z)}{z^p}\right)^{\alpha}\varphi(z)\right), \tag{2.1}$$

where

$$\rho = \frac{1+|p\beta+\gamma-1|^2-|1-(p\beta+\gamma-1)^2|}{4\Re\{p\beta+\gamma-1\}} \quad (\Re\{p\beta+\gamma-1\}>0). \tag{2.2}$$

Then the subordination:

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{g(z)}{z^p}\right)^{\alpha}\varphi(z), \tag{2.3}$$

implies that

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g)(z)}{z^p}\right)^{\beta}\phi(z), \tag{2.4}$$

where $I_{p;\alpha,\beta,\gamma,\delta}^{\phi,\varphi}$ *is the integral operator defined by* (1.3). *Moreover, the function*

$$z\left(\frac{I_{p;\alpha,\beta,\gamma,\delta}^{\phi,\varphi}(g)(z)}{z^p}\right)^{\beta}\phi(z)$$

is the best dominant.

Remark 2.1

We note that ρ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \rho \leq 1/2$.

Next, we prove a dual Problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2

Let $f, g \in \mathcal{A}_{\varphi;p,\alpha,\delta}$*, where* $\mathcal{A}_{\varphi;p,\alpha,\delta}$ *is defined by* (1.5)*. Suppose also that*

$$\Re\left\{1 + \frac{z\nu''(z)}{\nu'(z)}\right\} > -\rho \quad \left(z \in \mathbb{U};\ \nu(z) := z\left(\frac{g(z)}{z^p}\right)^{\alpha} \varphi(z)\right),$$

where ρ *is given by* (2.2)*. If* $z\left(f(z)/z^p\right)^{\alpha}\varphi(z)$ *is univalent in* $\mathbb{U}$ *and* $z\left(I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)/z^p\right)^{\beta}\phi(z) \in \mathcal{Q}$*, where* $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}$ *is the integral operator defined by* (1.3)*, then the superordination:*

$$z\left(\frac{g(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z) \tag{2.5}$$

implies that

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\phi(z).$$

Moreover, the function

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g)(z)}{z^p}\right)^{\beta}\phi(z)$$

is the best subordinant.

If we combine Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type result.

Theorem 2.3

Let $f, g_k \in \mathcal{A}_{\varphi;p,\alpha,\delta}(k=1,2)$, *where* $\mathcal{A}_{\varphi;p,\alpha,\delta}$ *is defined by* (1.5). *Suppose also that*

$$\Re\left\{1+\frac{z\nu_k''(z)}{\nu_k'(z)}\right\} > -\rho \quad \left(z\in\mathbb{U};\ \nu_k(z):=z\left(\frac{g_k(z)}{z^p}\right)^{\alpha}\varphi(z);\ k=1,2\right), \tag{2.6}$$

where ρ *is given by* (2.2). *If* $z\left(f(z)/z^p\right)^{\alpha}\varphi(z)$ *is univalent in* $\mathbb{U}$ *and* $z\left(I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)/z^p\right)^{\beta}\phi(z)\in\mathcal{Q}$, *where* $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}$ *is the integral operator defined by* (1.3), *then the subordination relation*:

$$z\left(\frac{g_1(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{g_2(z)}{z^p}\right)^{\alpha}\varphi(z)$$

implies that

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta}\phi(z).$$

Moreover, the functions

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \quad \text{and} \quad z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

Remark 2.2

If we take the parameters α, β, γ and δ with the restrictions $p = 1$, $\phi(z) = \varphi(z) = 1$, $\alpha = \beta$, $\gamma = \delta$ in Theorems 2.1, 2.2 and 2.3, then we have the results obtained by Owa and Srivastava [17] and Cho and Srivastava [5].

The assumption of Theorem 2.3, that the functions

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha} \quad \text{and} \quad z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\phi(z)$$

need to be univalent in $\mathbb{U}$, will be replaced by another conditions in the following result.

[17] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39**(1987), 1057-1077.

[5] Nak Eun Cho and H.M. Srivastava, A class of nonlinear integral operators preserving subordination and superordination, **18**(2007), 445–454.

Corollary 2.1

Let $f, g_k \in \mathcal{A}_{\varphi;p,\alpha,\delta}(k=1,2)$*, where* $\mathcal{A}_{\varphi;p,\alpha,\delta}$ *is defined by* (1.5). *Suppose also that the condition* (2.6) *is satisfied and*

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\rho \quad \left(z\in\mathbb{U};\ \psi(z) := z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z);\ f\in\mathcal{Q}\right), \quad (2.7)$$

where ρ *is given by* (2.2). *Then the subordination relation*:

$$z\left(\frac{g_1(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{g_2(z)}{z^p}\right)^{\alpha}\varphi(z)$$

implies that

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(f)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta}\phi(z),$$

where $I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}$ *is the integral operator defined by* (1.3). *Moreover, the functions*

$$z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \quad and \quad z\left(\frac{I^{\phi,\varphi}_{p;\alpha,\beta,\gamma,\delta}(g_2)(z)}{z^p}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

Taking $p\beta+\gamma-1=1$ in Theorem 2.3, we have the following.

Corollary 2.2

Let $f, g_k \in \mathcal{A}_{\varphi;p,\alpha,2-p\alpha}(k=1,2)$, *where* $\mathcal{A}_{\varphi;p,\alpha,2-p\alpha}$ *is defined by* (1.5) *with* $\delta=2-p\alpha$. *Suppose that*

$$\Re\left\{1+\frac{z\nu_k''(z)}{\nu_k'(z)}\right\}>-\frac{1}{2}\quad \left(z\in\mathbb{U};\ \nu_k(z):=z\left(g_k(z)/z^p\right)^{\alpha}\varphi(z));\ k=1,2\right).$$

If $z(f(z)/z^p)^{\alpha}\varphi(z)$ *is univalent functions in* $\mathbb{U}$ *and* $z(I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}f(z)/z^p)^{\beta}\phi(z)\in\mathcal{Q}$, *where the integral operator* $I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}$ *is defined by* (1.3) *with* $\gamma=2-p\beta$ *and* $\delta=2-p\alpha$, *then the subordination relation*:

$$z\left(\frac{g_1(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{f(z)}{z^p}\right)^{\alpha}\varphi(z) \prec z\left(\frac{g_2(z)}{z^p}\right)^{\alpha}\varphi(z)$$

implies that

$$z\left(\frac{I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \prec z\left(\frac{I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}(f)(z)}{z^p}\right)^{\beta}\phi(z)$$
$$\prec z\left(\frac{I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}(g_2)(z)}{z^p}\right)^{\beta}\phi(z). \quad (2.8)$$

Moreover, the functions

$$z\left(\frac{I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}(g_1)(z)}{z^p}\right)^{\beta}\phi(z) \text{ and } z\left(\frac{I^{\phi,\varphi}_{p;p\alpha,p\beta,2-p\beta,2-p\alpha}(g_2)(z)}{z^p}\right)^{\beta}\phi(z)$$

are the best subordinant and the best dominant, respectively.

Thank you for your attention!