Title: Existence and exponential estimates of the solution to stochastic differential equations.

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Abstract: In this presentation, we first introduce the basic concepts used in stochastic differential equations.

This main aim of this presentation investigates the existence and uniqueness of the solutions to stochastic differential equations under special conditions.

In addition, we establish the exponential estimation of the solution for the equations.

■ The questions regarding stochastic differential equations:
∘ What is the solution?
∘ If there is a solution, is it unique?
∘ What are the properties of the solution?
∘ How can the solution be obtained in practice?

■ Let
⋄ $(\Omega, \mathcal{F}, P)$ be a probability space.
⋄ $B(t) = (B_1(t), \cdot, B_m(t))^T$ be an m-dimensional Brownian motion.
⋄ X_0 be an $\mathcal{F}_{t_0}$- measurable (where $0 \le t_0 < T < \infty$) $\mathcal{R}^d$-valued random variable such that $E|X_0|^2 < \infty$
⋄ $f : R^d \times [t_0, T] \to R^d$ and $g : R^d \times [t_0, T] \to R^{d \times m}$ be Borel measurable.

Definition 1 (SDEs solution) *We say that the stochastic process* $\{X(t)\}_{t_0 \le t \le T}$ *is a solution of the stochastic differential equation*

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t) \quad (1)$$

with initial condition $X(t_0) = X_0$ *if the following conditions hold:*
(i) $\{X(t)\}$ *is continuous and* $\mathcal{F}_t$*-adapted;*
(ii) $\{f(X(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ *and* $\{g(X(t), t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$*;*
(iii) the integral equation

$$X(t) = X_0 + \int_{t_0}^{t} f(X(s), s)ds + \int_{t_0}^{t} g(X(s), s)dB(s), \quad (2)$$

hold for every $t \in [t_0, T]$ *with probability 1.*

A solution $\{X(t)\}$ *is said to be unique if any other solution* $\{\overline{X}(t)\}$ *is indisinguishable with* $\{X(t)\}$*, that*

$$P\{X(t) = \overline{X}(t),\ \text{for all}\ \ t_0 \le t \le T\} = 1.$$

The following theorems are conditions to ensure the existence and uniqueness of the solution to the stochastic differential equation.

Theorem 2 *([A3], Mao)*
Assume that there exist two positive constants $\overline{K}$ *and K such that*
(i) (Lipschitz condition) for all $x, \overline{x} \in R^d$ *and* $t \in [t_0, T]$

$$|f(x,t) - f(\overline{x},t)|^2 \vee |g(x,t) - g(\overline{x},t)|^2 \leq \overline{K}|x - \overline{x}|^2$$

(ii) (Linear growth condition) for all $(x,t) \in R^d \times [t_0, T]$

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq K(1 + |x|^2)$$

Then there exists a unique solution $x(t)$ *to equation (1) and the solution belongs to* $\mathcal{M}^2([t_0, T]; R^d)$.

Theorem 3 *Assume that there exist positive number K such that*

(i) For all $x, y \in R^d$, $t \in [t_0, T]$, and $0 < \alpha \leq 1$ it follows that

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq \kappa\ (|x-y|^{2\alpha}), \qquad (3)$$

where $\kappa(\cdot)$ is a concave non-decreasing function from R_+ to R_+ such that $\kappa(0) = 0$, $\kappa(u) > 0$, for $u > 0$ and $\int_{0+} \frac{1}{\kappa(u)} du = \infty$.
(ii) For all $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in R^d \times [t_0, T]$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \qquad (4)$$

Then there exists a unique solution $x(t)$ to equation (1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$.

■ To demonstrate the generality of our results, let us illustrate it using a concave function $\kappa(\cdot)$. Let $K > 0$ and let $\delta \in (0,1)$ be sufficiently small. Define

$$\kappa_1(u) = Ku \qquad ,u > 0$$

$$\kappa_2(u) = \begin{cases} u\log(u^{-1}) & ,0 \le u < \delta \\ \delta\log(\delta^{-1}) + \dot{\kappa_2}(\delta-)(u-\delta) & ,u > \delta \end{cases}$$

$$\kappa_3(u) = \begin{cases} u\log(u^{-1})\log\log(u^{-1}) & ,0 \le u < \delta \\ \delta\log(\delta^{-1})\log\log(\delta^{-1}) + \dot{\kappa_3}(\delta-)(u-\delta) & ,u > \delta \end{cases}$$

They are all concave nondecreasing functions satisfying $\kappa_i(u) > 0$, for $u > 0$ and $\int_{0+} \frac{1}{\kappa_i(u)} du = \infty$. In particular, we see that Lipschitz condition is a special case of our proposed condition (3).

Theorem 4 *Let $p \geq 2$ and $x_0 \in \mathcal{L}^p\left(\Omega; R^d\right)$. Assume that conditions (3) and (4) hold. Then*

$$E \sup_{t_0 \leq t \leq T} |x(t)|^p \leq C_1 \left(1 - 3^{\frac{p-2}{2}} b^{\frac{p}{2}} C_{10} C_{11}^{\alpha-1} (\alpha - 1)(T - t_0)\right)^{\frac{1}{1-\alpha}} := C_2,$$

where $C_0 = 2^{\frac{p}{2}} 3^{p-1} \left[(T - t_0)^{p-1} + \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} (T - t_0)^{\frac{p-2}{p}} \right]$

and $C_1 = 3^{p-1} E|x_0|^p + 3^{\frac{p-2}{2}} C_0 (T - t_0)(a^{\frac{p}{2}} + K^{\frac{p}{2}})$.

The following theorem aims to establish new results for the L^p-continuity of the solution for SDEs.

Theorem 5 *Let $p \geq 2$ and $x_0 \in \mathcal{L}^p\left(\Omega; R^d\right)$. Assume that conditions (3) and (4) hold. Then*

$$E|x(t) - x(s)|^p \leq 3^{\frac{p-2}{2}} C_3 [a^{\frac{p}{2}} + K^{\frac{p}{2}} + b^{\frac{p}{2}} C_2^{\alpha}](t-s)^{\frac{p}{2}},$$

for all $t_0 \leq s < t \leq T$, where $C_3 = 2^{\frac{p}{2}}[2^{p-1}(T-t_0)^{\frac{p}{2}} + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}]$.

■ Proof of Theorem 5

By the elemantary inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we obtatin that

$$\begin{aligned} &E|x(t) - x(s)|^p \\ &\leq 2^{p-1} E\left|\int_s^t f(x(r), r)dr\right|^p + 2^{p-1} E\left|\int_s^t g(x(r), r)dB(r)\right|^p. \end{aligned}$$

Using Hölder's inequality, conditions (3) and (4), we have that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\leq [2(t-s)]^{p-1} E\int_s^t [2|f(x(r),r)-f(0,r)|^2 + 2|f(0,r)|^2]^{\frac{p}{2}} dr \\
&\leq C_3 (t-s)^{\frac{p-2}{2}} E\int_s^t [\kappa(|x(r)|^{2\alpha}) + K]^{\frac{p}{2}} dr,
\end{aligned}
$$

where $C_3 = 2^{\frac{p}{2}}[2^{p-1}(T-t_0)^{\frac{p}{2}} + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}]$.

Given that $\kappa(\cdot)$ is concave function and $\kappa(0)=0$, we can find a pair of positive constants a and b such that $\kappa(u) \leq a + bu$ for all $u \geq 0$. Therefore

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\leq C_3 (t-s)^{\frac{p-2}{2}} E\int_s^t [a + b|x(r)|^{2\alpha} + K]^{\frac{p}{2}} dr.
\end{aligned}
$$

By the elemantary inequality $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$, we obtatin that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\leq 3^{\frac{p-2}{2}} C_3[a^{\frac{p}{2}} + K^{\frac{p}{2}}](t-s)^{\frac{p}{2}} \\
&\quad +3^{\frac{p-2}{2}} C_3 b^{\frac{p}{2}} (t-s)^{\frac{p-2}{2}} E\int_s^t [E \sup_{t_0 \leq u \leq r} |x(u)|^p]^\alpha dr.
\end{aligned}
$$

By Theorem 4, we have that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\leq 3^{\frac{p-2}{2}} C_3[a^{\frac{p}{2}} + K^{\frac{p}{2}}](t-s)^{\frac{p}{2}} + 3^{\frac{p-2}{2}} C_3 b^{\frac{p}{2}} (t-s)^{\frac{p-2}{2}} E\int_s^t C_2^\alpha dr \\
&\leq 3^{\frac{p-2}{2}} C_3[a^{\frac{p}{2}} + K^{\frac{p}{2}} + b^{\frac{p}{2}} C_2^\alpha](t-s)^{\frac{p}{2}}.
\end{aligned}
$$

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