On Hilbert rings

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Properties of Hilbert ring

Remark

Let R be a commutative ring with identity. Then the following assertions hold.

1. An ideal I in a ring R is a G-ideal if and only if it is the contraction of a maximal ideal in the polynomial ring $R[X]$.
2. If R is a Hilbert ring such that every maximal ideal in R can be generated by k elements(k fixed), then any maximal ideal in $R[X_1, \ldots, X_n]$ can be generated by $k+n$ elements.
3. R is a Hilbert ring if and only if $R[X]$ is a Hilbert ring.

S-n-generated

Definition

Let R be a commutative ring with identity and S a (not necessarily saturated) multiplicative subset of R. An ideal I of R is *S-n-generated* if there exists $s \in S$ and $a_1, a_2, \ldots, a_n \in R$ such that $sI \subseteq (a_1, \ldots, a_n) \subseteq I$.

If an ideal I of R is S-n-generated, then I is a S-finite.

Recall that if every G-ideal in R is maximal, then we say R is a Hilbert ring.

Proposition

Let R be a Hilbert ring and S a multiplicative subset of R. Then the following assertions hold.

1. If every maximal ideal of R is S-n-generated, then every maximal ideal of $R[X]$ is S-$(n+1)$-generated.
2. If every maximal ideal of R is S-finite, then every maximal ideal of $R[X]$ is also S-finite.
3. Let R be a Hilbert ring, S a multiplicative subset of R and $X = \{X_1, \ldots, X_n\}$ a set of indeterminates over R. If every maximal ideal of R is S-finite, then every maximal ideal of $R[X]$ is also S-finite. More precisely, if every maximal ideal of R is S-m-generated, then every maximal ideal of $R[X]$ is S-$(m+n)$-generated.

Proof

(1) Let M be a maximal ideal of $R[X]$. Then $M \cap R$ is a G-ideal of R. Since R is a Hilbert ring, $M \cap R$ is a maximal ideal of R; so by the assumption, there exist $s \in S$ and $a_1, \ldots, a_n \in R$ such that

$$s(M \cap R) \subseteq (a_1, \ldots, a_n) \subseteq M \cap R.$$

Note that $M = (M \cap R)R[X] + (f)$ for some $f \in R[X]$. Hence we obtain

$$sM = s((M \cap R) + (f)) \subseteq (a_1, \ldots, a_n, f) \subseteq M.$$

Thus M is an S-$(n+1)$-generated.
(2) and (3) follows from (1).

Example and Counter example

In general, if every maximal ideal of R is S-principal, then every maximal ideal of $R[X]$ is not S-principal.

Example

Let R be a PID and let S be a multiplicative subset of R.
Then R is a Hilbert ring.
Let M be a maximal ideal in $R[X]$.
Then $P = M \cap R$ is a G-ideal, hence $P = (p)$ for some $p \in R$.
Thus $M = (p, f)$ for some monic polynomial $f \in R[X]$.
Case 1. $M \cap S \neq \emptyset$.
Let $s \in M \cap S$. Then $sM \subseteq (s) \subseteq M$. Hence M is S-principal.

Example and Counter example

Counter example

Case 2. $M \cap S = \emptyset$.
Suppose that there is a principal ideal (g) such that $sM \subseteq (g) \subseteq M$ for some $s \in S$. Since $sp \in (g)$, we have $g \in R$.
Then $g = pr$ for some $r \in R$. Hence $sf \in (p)$.
Since (p) is a prime ideal of $R[X]$, $s \in (p)$ or $f \in (p)$.
Since $M \cap S = \emptyset$, $f \in (p)$, a contradiction.
Therefore M is not S-principal.

Recall that an element of the *semigroup ring* $R[\Gamma]$ is of the form $a_0 + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n}$ where $a_0, \ldots, a_n \in R$ and $\alpha_1, \ldots, \alpha_n \in \Gamma$. In fact, $R[\Gamma]$ is a ring and we say $R[\Gamma]$ is a semigroup ring.

Proposition

Let R be a commutative ring with identity and let Γ be a finitely generated semigroup which satisfying well ordering property. Then an ideal in R is a G-ideal if and only if it is the contraction of a maximal ideal in semigroup ring $R[\Gamma]$.

Corollary

Let R be a commutative ring with identity and let Γ be a numerical semigroup. Then an ideal in R is a G-ideal if and only if it is the contraction of a maximal ideal in semigroup ring $R[\Gamma]$.

Proof (if)

Since Γ be a finitely generated semigroup, $\Gamma = \langle \alpha_1, \ldots, \alpha_n \rangle$ for some $\alpha_1, \ldots, \alpha_n \in \Gamma$.
Let M be a maximal ideal of $R[\Gamma]$ and let $P = M \cap R$
Consider $\varphi : R/P \to R[\Gamma]/M$ given by $r + P \mapsto r + M$.
Then φ is a monomorphism and $R/P \cong \varphi(R/P)$.
Note that $R[\Gamma]/M = \varphi(R/P)[X^{\alpha_1} + M, \ldots, X^{\alpha_n} + M]$.
Hence R/P is a G-domain.
Thus P is a G-ideal.

Proof (Only if)

Conversely let I be a G-ideal. Then R/I is a G-domain.
So there is a nonzero element $u \in R/I$ such that $(R/I)[u^{-1}]$ is a quotient field K of R/I.
Let $\varphi : R[\Gamma] \to K$ given by $\sum_{i=0}^{n} r_iX^{\beta_i} \mapsto r_0 + I$ where $\beta_i \in \Gamma$.
Then φ is an epimorphism.
Now we claim that $\ker \varphi = (I, X^{\alpha_1}, \ldots, X^{\alpha_n}) = M$.
Let $f = \sum_{i=0}^{n} r_iX^{\beta_i} \in R[\Gamma]$.
If $f \in \ker \varphi$, then $r_0 \in I$. Hence $f \in M$.
If $f \in M$, then $r_0 \in I$, hence $f \in \ker \varphi$.
By first isomorphism theorem, $R[\Gamma]/M \cong K$.
Hence $(I, X^{\alpha_1}, \ldots, X^{\alpha_n})$ is a maximal ideal in $R[\Gamma]$.
Since $(I, X^{\alpha_1}, \ldots, X^{\alpha_n}) \cap R = I$, every G-ideal is the contraction of a maximal ideal in $R[\Gamma]$.

Theorem

Let R be a commutative ring with identity and let Γ be a finitely generated semigroup. Then R is a Hilbert ring if and only if $R[\Gamma]$ is a Hilbert ring.

Corollary

Let R be a commutative ring with identity and let Γ be a numerical semigroup. Then R is a Hilbert ring if and only if $R[\Gamma]$ is a Hilbert ring.

Proof

Since Γ is finitely generated, $\Gamma = \langle \alpha_1, \ldots, \alpha_n \rangle$ for some $\alpha_1, \ldots, \alpha_n \in \Gamma$.
The "if" part is clear.
For the converse, suppose that $R[\Gamma]$ is not a Hilbert ring.
Then there exists G-ideal M in $R[\Gamma]$ such that M is not a maximal ideal. Let $P = M \cap R$.
Consider $\varphi : R/P \to R[\Gamma]/M$ given by $r + P \mapsto r + M$.
Then $R[\Gamma]/M \cong (R/P)[X_{\alpha_1}, \ldots, X_{\alpha_n}]$ where $X_{\alpha_i} = X^{\alpha_i} + M$.
Hence R/P is a G-domain and X_{α_i} is algebraic over a G-domain $(R/P)[X_{\alpha_1}, \ldots, X_{\alpha_{i-1}}]$ where $1 \leq i \leq n$.
By the assumption, $(R/P)[X_{\alpha_1}, \ldots, X_{\alpha_n}]$ is a field.
Hence $R[\Gamma]/M$ is a field, a contradiction.

Remark

Let $A \subseteq B$ be a ring extension. Then the following assertions holds.

1. If B is integral over A, then A is a Hilbert ring if and only if B is a Hilbert ring.
2. $A + XB[X]$ is a Hilbert ring if and only if A and B are Hilbert rings.

Proposition

Let $A \subseteq B$ be a ring extension and let Γ be a numerical semigroup. The following statements are equivalent.

1. $A + B[\Gamma^*]$ is a Hilbert ring.
2. A and B are Hilbert rings.

Proof

Note that $A + XB[X]$ is integral over $A + B[\Gamma^*]$.
If A and B be Hilbert rings, then $A + XB[X]$ is a Hilbert ring.
Hence $A + B[\Gamma^*]$ is a Hilbert ring.
Conversely if $A + B[\Gamma^*]$ be a Hilbert ring,
then $A + XB[X]$ is a Hilbert ring.
Hence A and B are Hilbert rings.

Thank you for your attention!