Title: An Existence and Uniqueness Theorem of Stochastic Differential Equations and the Properties of Their Solution

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Abstract

Abstract: In this presentation, we first introduce definition of solution of the stochastic differential equation.

We show the existence and uniqueness of solution to stochastic differential equations under weakened Hölder condition and a weakened linear growth condition.

Furthermore, the properties of their solutions investigated and estimate for the error between Picard iterations $x_n(t)$ and the unique solution $x(t)$ of stochastic differential equations.

■ **We study existence and uniqueness of solutions to a stochastic differential equation:**

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T]$$

with initial value $x(t_0) = x_0$, where $0 \le t_0 < T < \infty$.

■ **Some of the main (mathematical) questions regarding such equation:**

(i) Is there a solution?

(ii) If there is a solution, is it unique?

(iii) What kind of properties do solutions have?

Definition 1 (SDE solution) *We say that the stochastic process* $\{x(t)\}_{t_0 \le t \le T}$ *is a solution of the stochastic differential equation*

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{1}$$

with initial condition $x(t_0) = x_0$ *if the following conditions hold:*
(i) $\{x(t)\}$ *is continuous and* $\mathcal{F}_t$*-adapted;*
(ii) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ *and* $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^{d \times m})$;
(iii) the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s),$$

hold for every $t \in [t_0, T]$ *with probability 1.*

Definition 2 (Solution uniqueness) *A solution* $\{x(t)\}$ *is said to be unique if any other solution* $\{\overline{x}(t)\}$ *is indisinguishable with* $\{x(t)\}$*, that*

$$P\{x(t) = \overline{x}(t), \text{ for all } t_0 \le t \le T\} = 1.$$

In order to obtain the existence of solutions to SDEs, let $x_0(t) = x_0$ for $t_0 \le t \le T$. For each $n = 1, 2, \cdots$, and define Picard sequence

$$x_n(t) = x_0 + \int_{t_0}^{t} f\left(x_{n-1}(s), s\right) ds + \int_{t_0}^{t} g\left(x_{n-1}(s), s\right) dB(s). \quad (2)$$

We give the existence and uniqueness theorem to the solution of equation (1) by approximate solutions by means of Picard sequence.

Theorem 3 *([A2], Mao) Assume that there exist two positive constants $\overline{K}$ and K such that*
(i) (Lipschitz condition) for all $x, \overline{x} \in R^d$ and $t \in [t_0, T]$

$$|f(x,t) - f(\overline{x},t)|^2 \vee |g(x,t) - g(\overline{x},t)|^2 \le \overline{K}|x - \overline{x}|^2$$

(ii) (Linear growth condition) for all $(x,t) \in R^d \times [t_0, T]$

$$|f(x,t)|^2 \vee |g(x,t)|^2 \le K(1 + |x|^2)$$

Then there exists a unique solution $x(t)$ to equation (1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$.

Theorem 4 (Existence and uniqueness of solution) *Assume that there exist two positive constants* $\overline{K}$ *and K such that*
(H1) (Weakened Hölder condition) For any $x, y \in R^d$ *and* $t \in [t_0, T]$*, it follows that*

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq \overline{K}|x-y|^{2\alpha},$$

where $0 < \alpha \leq 1$ *is a constant.*

(H2) (Weakened linear growth condtion) For any $t \in [t_0, T]$ *it follows that* $f(0,t), g(0,t) \in \mathcal{L}^2([t_0, T])$ *it follows that*

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K,$$

Then there exists a unique solution to the SDEs (1). Moreover, the solution belongs to $\mathcal{M}^2\left([t_0, T]; R^d\right)$*.*

■ We prepare two lemmas in order to prove this theorem.

Lemma 5 *Let $u(t)$ and $a(t)$ be continuous functions on $[0,T]$. Let $k \geq 1$ and $0 < p \leq 1$ be constants. If $u(t) \leq k + \int_{t_0}^{t} a(s)u^p(s)ds$ for $t \in [t_0, T]$ then*

$$u(t) \leq k \exp\left(\int_{t_0}^{t} a(s)ds\right)$$

for $t \in [t_0, T]$

Lemma 6 *Let the assumption (H1) and (H2) hold. If $x(t)$ is a solution of (1), then*

$$E\left(\sup_{t_0 \leq t \leq T} |x(t)|^2\right) \leq C \exp\left(6(T - t_0 + 4)\overline{K}(T - t_0)\right),$$

where $C = 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0)$ with $C \geq 1$.

■ **Proof.** For each number $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x(t)| \geq n\}.$$

Let $x_n(t) = x(t \wedge \tau_n)$, $t \in [t_0, T]$.Then $x_n(t)$ satisfies the following equation

$$x_n(t) = x_0 + \int_{t_0}^{t} f\left(x_n(s), s\right) I_{[t_0,\tau_n]}(s) ds + \int_{t_0}^{t} g\left(x_n(s), s\right) I_{[t_0,\tau_n]}(s) dB(s).$$

Using the elementary $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the Hölder inequality, condition (H1) and (H2), one can show that

$$E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \leq C + 6(T - t_0 + 4)\overline{K} \int_{t_0}^{t} E\left(\sup_{t_0 \leq r \leq s} |x_n(r)|^{2\alpha}\right) ds,$$

where $C = 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0)$. By the Lemma 5

$$E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \leq C \exp\left(6(T - t_0 + 4)\overline{K}(T - t_0)\right)$$

with $C \geq 1$. Consequently the required inequality follows by letting $n \to \infty$.

■ **Proof of Theorem 4 : Uniqueness** Let $x(t)$ and $\overline{x}(t)$ be two solutions. By the Lemma 6, $x(t)$, $\overline{x}(t) \in \mathcal{M}^2\left([t_0, T]; R^d\right)$. Note that

$$x(t) - \overline{x}(t) \\ = \int_{t_0}^{t} [f(x(s), s) - f(\overline{x}(s), s)]ds + \int_{t_0}^{t} [g(x(s), s) - g(\overline{x}(s), s)]dB(s).$$

Using the Hölder inequality, Moment inequality and (H1), for any $\varepsilon > 0$, one can show that

$$E\left(\sup_{t_0 \le s \le t} |x(s) - \overline{x}(s)|^2\right) \\ \le \varepsilon + 2\overline{K}(T - t_0 + 4)\int_{t_0}^{t} E \sup_{t_0 \le r \le s} |x(r) - \overline{x}(r)|^{2\alpha} ds.$$

Therefore, by the Stachurska's inequality, we have

$$E\left(\sup_{t_0 \le s \le t} |x(s) - \overline{x}(s)|^2\right) = 0.$$

■ **Proof of Theorem 4 : Existence** From Picard sequence (2), we have $x_0 \in \mathcal{M}^2\left([t_0, T]; R^d\right)$. By induction $x_n(t) \in \mathcal{M}^2\left([t_0, T]; R^d\right)$, in fact

$$E\left(\sup_{t_0 \le s \le t} |x_n(s)|^2\right) \le C + 6(T - t_0 + 4)\overline{K} \int_{t_0}^{t} E\left(\sup_{t_0 \le r \le s} |x_{n-1}(r)|^{2\alpha}\right) ds,$$

where $C = 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0)$. It follows note that for any $k \ge 1$,

$$\max_{1 \le n \le k} E\left(\sup_{t_0 \le s \le t} |x_n(s)|^2\right) \le \gamma \exp\left(6\overline{K}(T - t_0 + 4)(T - t_0)\right)$$

where $\gamma = C + 6(T - t_0 + 4)E|x_0|^{2\alpha}$. Since k is arbitrary, for all $n = 0, 1, 2, \cdots$, we deduce that

$$E\left(\sup_{t_0 \le s \le t} |x_n(s)|^2\right) \le \gamma \exp\left(6\overline{K}(T - t_0 + 4)(T - t_0)\right),$$

Next, we check that the sequence $\{x_n(t)\}$ is Cauchy sequence. For all $n \geq 0$ and $t_0 \leq t \leq T$, we have

$$\begin{aligned} & x_{n+1}(t) - x_n(t) \\ & = \int_{t_0}^{t} [f(x_n(s), s) - f(x_{n-1}(s), s)]ds \\ & + \int_{t_0}^{t} [g(x_n(s), s) - g(x_{n-1}(s), s)]dB(s). \end{aligned}$$

Using the Hölder inequality, Moment inequality and (H1), one can show that

$$\begin{aligned} & \max_{1 \leq n \leq k} E\left(\sup_{t_0 \leq s \leq t} |x_{n+1}(s) - x_n(s)|^2\right) \\ & \leq 2(T - t_0 + 4)\overline{K} \int_{t_0}^{t} \max_{1 \leq n \leq k} E\left(\sup_{t_0 \leq r \leq s} |x_{n+1}(r) - x_n(r)|^{2\alpha}\right) ds. \end{aligned}$$

Let $z(t) = \limsup_{n \to \infty} \max_{1 \leq n \leq k} E\left(\sup_{t_0 \leq s \leq t} |x_{n+1}(s) - x_n(s)|^2\right)$, we get

$$z(t) \leq 2(T - t_0 + 4)\overline{K} \int_{t_0}^{t} z^{\alpha}(s)ds.$$

By stachurska's inequality, we get $z(t) = 0$. This shows the sequence $\{x_n(t), n \geq 0\}$ is Cauchy sequence in $\mathcal{L}^2$. Hence, as $n \to \infty$, $x_n(t) \to x(t)$, that is $E|x_n(t) - x(t)|^2 \to 0$. Therefore, we obtain that $x(t) \in \mathcal{M}^2([t_0, t]; R^d)$. Now to show that $x(t)$ satisfy (1)

$$E\left|\int_{t_0}^{t}[f(x_n(s), s) - f(x(s), s)]ds + \int_{t_0}^{t}[g(x_n(s), s) - g(x(s), s)]dB(s)\right|^2$$

$$\leq 2(T - t_0 + 4)\int_{t_0}^{t} E\left(\sup_{t_0 \leq r \leq s}|x_n(r) - x(r)|^{2\alpha}\right)ds.$$

Noting that sequence $\{x_n(t)\}$ is uniformly converge on $[t_0, T]$, it means that

$$E\left(\sup_{t_0 \leq s \leq t}|x_n(s) - x(s)|^2\right) \to 0$$

as $n \to \infty$. Hence, taking limits on both sides in the Picard sequence, we obtain that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s). \quad on \quad t_0 \leq t \leq T.$$

Lemma 7 *Assume that (H1) and (H2) hold. Let $x_n(t)$ be the Picard iterations defined by (2). Then*

$$E\left(\sup_{t_0\le t\le T}|x_{n+1}(t)-x_n(t)|^2\right)\le (M(T-t_0))^{\frac{1-\alpha^n}{1-\alpha}}C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}$$

where $M=2\overline{K}(T-t_0+1)$.

Theorem 8 *Assume that (H1) and (H2) hold. Let $x(t)$ be the unique solution $x(t)$ of equation (1) and $x_n(t)$ be the Picard iteration defined by (2). Then*

$$E\left(\sup_{t_0\le t\le T}|x_n(t)-x(t)|^2\right)\le \gamma_1\exp\left(2M(T-t_0)\right)$$

for all $n\ge 1$, where $C=4(T-t_0+1)(T-t_0)(K+\overline{K}E|x_0|^2)$ and $M=2\overline{K}(T-t_0+1)$ and $\gamma_1=2\left(M(t-t_0)\right)^{\frac{1-\alpha^n}{1-\alpha}}C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}$.

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